CIE3109
Structural Mechanics 4

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Module : Unsymmetrical and/or
inhomogeneous cross section

LECTURE 3

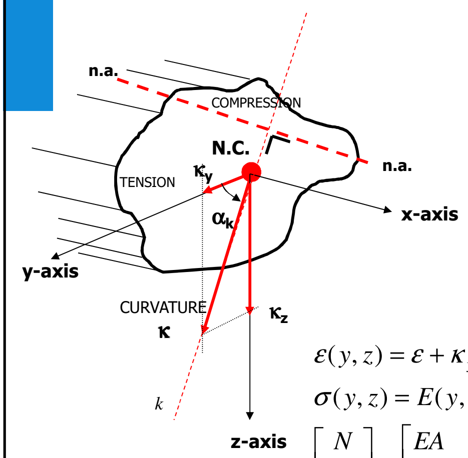
v2021

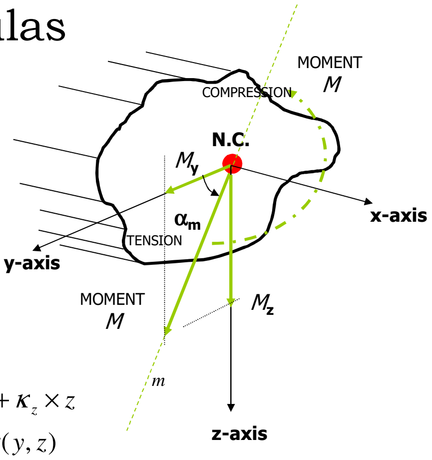
CIE3109 : Structural Mechanics 4

Lectures

- 1-2 Inhomogeneous and/or unsymmetrical cross sections
  - Introduction
  - General theory for extension and bending, beam theory
  - Unsymmetrical cross sections
    - Example curvature and loading
    - Example normalstress distribution
  - Deformations
- 3 Inhomogeneous cross sections
  - Refinement of the theory
  - Examples
- 4-5 Stresses and the core of the cross section
  - Normal stress in unsymmetrical cross section and the core
  - Shear stresses in unsymmetrical cross sections
  - Shear centre

SUMMARY of formulas





$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z$$

$$\sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & & \\ & EI_{yy} & EI_{yz} \\ & EI_{yz} & EI_{zz} \end{bmatrix} \times \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

CALCULATION SCHEME
Cross Section

- determine NC → - calculate EA, EI_{yy}, EI_{zz}, EI_{yz}

- determine the cross sectional forces N, M_y and M_z

↓

- compute the relevant deformation parameters $(\varepsilon, \kappa_y, \kappa_z)$ | **constitutive matrix**

↓

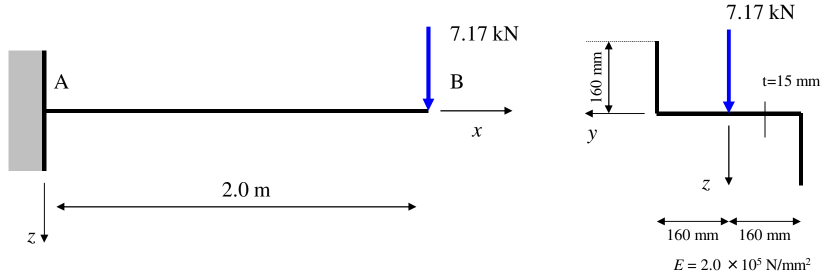
- calculate *strain distribution*
- calculate *stress distribution*

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & & \\ & EI_{yy} & EI_{yz} \\ & EI_{yz} & EI_{zz} \end{bmatrix} \times \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z$$

$$\sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

EXAMPLE 4: cantilever beam

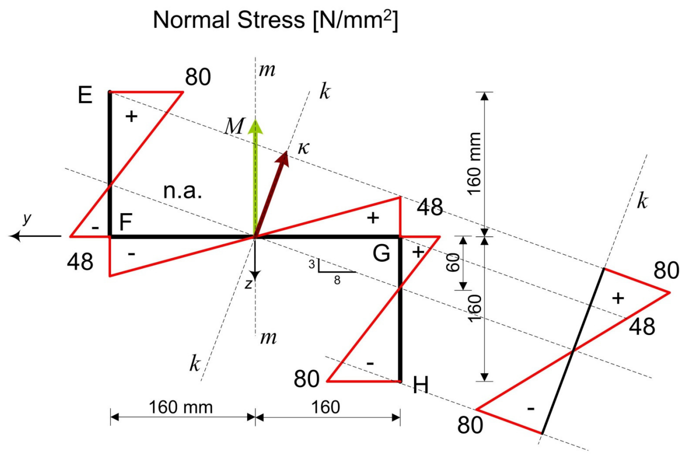


Questions : Let's demonstrate this on the whiteboard

- The normal stress distribution in the cross section at A.
- The displacement of point B.

(several methods both in y-z coordinate-system and in principal directions)

Example 4 : stress results



SUMMARY displacements

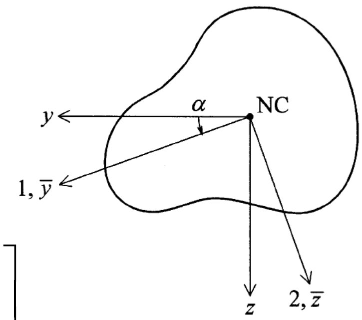
Original y-z coordinate system

- solve differential equations OR
- Use "pseudo" load for y- and z-direction and use this load in standard engineering equations OR
- Use curvature distribution in combination with moment area theoremes to obtain displacements and rotations (see notes)

Principal 1-2 coordinate system

- Use principal directions, decompose load in (1) and (2) direction, and compute displacements in (1) and (2) direction with standard engineering equations. Finally transformate the displacements back to y- and z-direction

PRINCIPAL DIRECTIONS



$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} EI_{\bar{y}\bar{y}} & 0 \\ 0 & EI_{\bar{z}\bar{z}} \end{bmatrix} \begin{bmatrix} \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix}$$

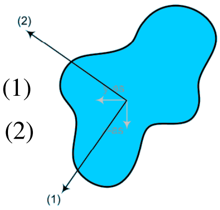
$$EI_{\bar{y}\bar{y}}, _{\bar{z}\bar{z}} = \tfrac{1}{2}\left(EI_{yy} + EI_{zz}\right) \pm \sqrt{\left(\tfrac{1}{2}(EI_{yy} - EI_{zz})\right)^2 + EI_{yz}^2}$$

$$\tan 2\alpha_{\bar{y}}, _{\bar{z}} = \frac{EI_{yz}}{\tfrac{1}{2}(EI_{yy} - EI_{zz})}$$

ROTATE TO PRINCIPAL DIRECTIONS

$\bar{z}$ = direction (1)

$\bar{y}$ = direction (2)



$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{\bar{y}\bar{y}} & 0 \\ 0 & 0 & EI_{\bar{z}\bar{z}} \end{bmatrix} \times \begin{bmatrix} \varepsilon \\ \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} \Rightarrow \begin{array}{l} \varepsilon = \dfrac{N}{EA} \\ \kappa_{\bar{y}} = \dfrac{M_{\bar{y}}}{EI_{\bar{y}\bar{y}}} \\ \kappa_{\bar{z}} = \dfrac{M_{\bar{z}}}{EI_{\bar{z}\bar{z}}} \end{array}$$

fully uncoupled !

Consequences for strains & stresses only in principal directions

$$\sigma(\bar{y},\bar{z}) = E(\bar{y},\bar{z}) \times \varepsilon(\bar{y},\bar{z})$$

$$\varepsilon(\bar{y},\bar{z}) = \varepsilon + \kappa_{\bar{y}} \times \bar{y} + \kappa_{\bar{z}} \times \bar{z}$$

THUS..............

$$\sigma(\bar{y},\bar{z}) = E(\bar{y},\bar{z}) \left[\frac{N}{EA} + \frac{M_{\bar{y}} \times \bar{y}}{EI_{\bar{y}\bar{y}}} + \frac{M_{\bar{z}} \times \bar{z}}{EI_{\bar{z}\bar{z}}} \right]$$

Homogeneous : $\sigma(\bar{y},\bar{z}) = \dfrac{N}{A} + \dfrac{M_{\bar{y}} \times \bar{y}}{I_{\bar{y}\bar{y}}} + \dfrac{M_{\bar{z}} \times \bar{z}}{I_{\bar{z}\bar{z}}} = \dfrac{N}{A} + \dfrac{M_{\bar{y}}}{W_{\bar{y}\bar{y}}} + \dfrac{M_{\bar{z}}}{W_{\bar{z}\bar{z}}}$ for most outer fibres

Flow of actions:

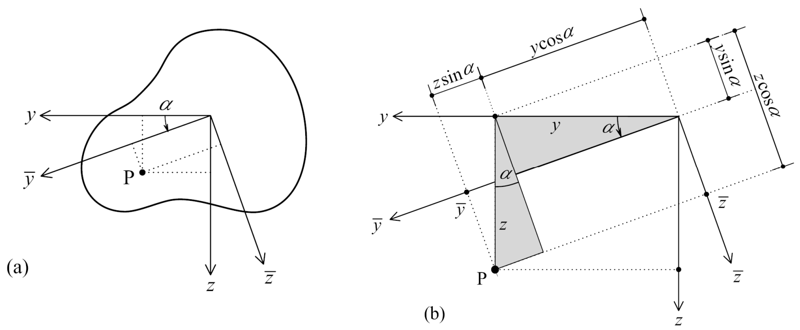
stress

- Find all cross sectional properties
- Determine the principal axes of the cross section
  → use Mohr's circle or transformation rules (for 2nd order tensor)
- Resolve the forces My and Mz in the principal directions
  → use coordinate-transformation rules
- Compute the distance of a certain fibre to the NC in the principal coordinate system
  → use coordinate-transformation rules
- Determine the stress in the fibre with the stress formula on the previous slide

dispacement

- Find the displacements in de principal directions (forget me not's)
- Resolve the displacements in the original coordinate system
  → use coordinate-transformation rules (inverse)

Coordinate rotations
first order tensor transformation



$$\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \qquad \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix}$$

Example 4 : Cross Sectional Data (principal direction approach)

slide 8

principal values :

$$\tan(2\alpha) = -1$$

$$I_1 = (\tfrac{5}{3} + \sqrt{2})ta^3$$

$$I_2 = (\tfrac{5}{3} - \sqrt{2})ta^3$$

rotate coordinate system by -22.5°

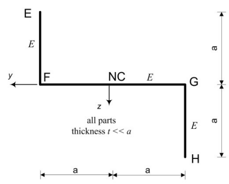
$$I_{\bar{y}\bar{y}} = I_1 = (\tfrac{5}{3} + \sqrt{2})ta^3 = 189.3 \times 10^6 \text{ mm}^4$$

$$I_{\bar{z}\bar{z}} = I_2 = (\tfrac{5}{3} - \sqrt{2})ta^3 = 15.5 \times 10^6 \text{ mm}^4$$

$$M_{\bar{y}} = M_y \cos(-22.5) + M_z \sin(-22.5) = 5.49 \times 10^6 \text{ Nmm}$$

$$M_{\bar{z}} = -M_y \sin(-22.5) + M_z \cos(-22.5) = -13.25 \times 10^6 \text{ Nmm}$$

Example 4 : stresses



Location of points E-F-G-H in rotated coordinate system (in rad):
(principal directions)

Point	y	z	$\bar{y}$	$\bar{z}$		
					alpha	-0.3927
E	160	-160	209.0501	-86.591376	Cos	0.92388
F	160	0	147.8207	61.229349	Sin	-0.38268
G	-160	0	-147.821	-61.229349		
H	-160	160	-209.05	86.591376		

σ at E : 80 N/mm²
see also slide 6

$$\sigma(\bar{y},\bar{z}) = \frac{M_{\bar{y}}\,\bar{y}}{I_{\bar{y}\bar{y}}} + \frac{M_{\bar{z}}\,\bar{z}}{I_{\bar{z}\bar{z}}}$$

Example : displacements (principal direction approach)

$$F_{\bar{y}} = F_y \cos(-22.5) + F_z \sin(-22.5) = -2.744\times10^3 \text{ N} \qquad \text{with: } F_y = 0;$$

$$F_{\bar{z}} = -F_y \sin(-22.5) + F_z \cos(-22.5) = 6.624\times10^3 \text{ N} \qquad F_z = 7.17\times10^3 \text{ N}$$

$$u_{\bar{y}} = \frac{F_{\bar{y}} l^3}{3EI_{\bar{y}\bar{y}}} = \frac{-2.744\times10^3\times2000^3}{3\times2.0\times10^5\times189.3\times10^6} = -0.1933 \quad \text{mm}$$

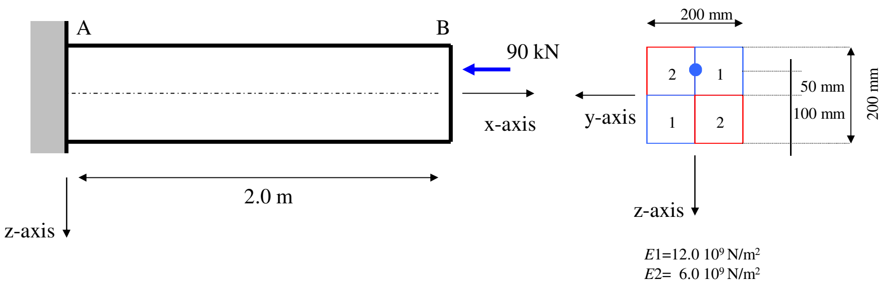
$$u_{\bar{z}} = \frac{F_{\bar{z}} l^3}{3EI_{\bar{z}\bar{z}}} = \frac{6.624\times10^3\times2000^3}{3\times2.0\times10^5\times15.5\times10^6} = 5.69 \quad \text{mm}$$

in principal coordinate system

$$u_y = u_{\bar{y}} \cos(-22.5) - u_{\bar{z}} \sin(-22.5) = 2.00 \text{ mm}$$

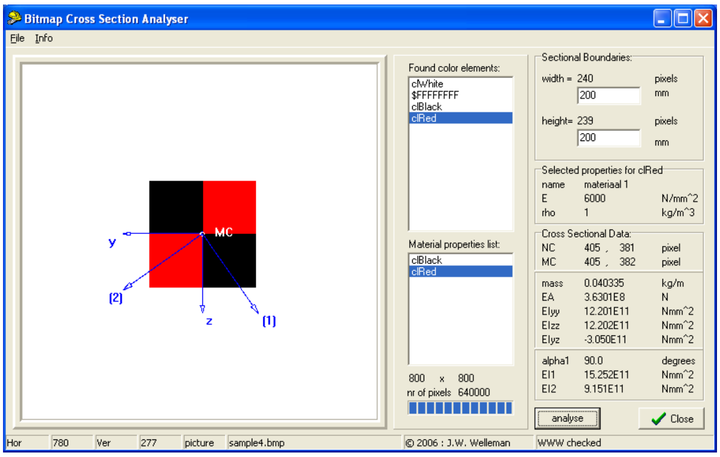
$$u_z = u_{\bar{y}} \sin(-22.5) + u_{\bar{z}} \cos(-22.5) = 5.33 \text{ mm}$$

EXAMPLE 5 : inhomeogenous

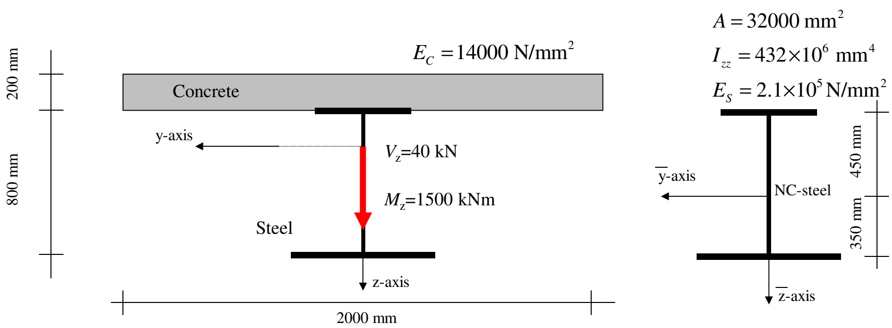


- Find the stress distribution in the cross section at A.
- Find the displacement of point B.

Playstation ..



EXAMPLE 6 : symmetrical and inhomogeneous



- Find the normal stress distribution over the depth of the cross section